Kerr-stabilized super-resonant modes in coupled-resonator optical waveguides

Shayan Mookherjea*
Department of Electrical Engineering, 136-93 California Institute of Technology, Pasadena, California 91125

Amnon Yariv
Department of Applied Physics, 128-95 California Institute of Technology, Pasadena, California 91125
(Received 8 May 2002; published 15 October 2002)

We investigate the effects of the optical Kerr nonlinearity in a coupled-resonator optical waveguide (CROW). Under certain conditions, there exists a stationary spatial distribution of the field whose envelope does not change with time—a super-resonant mode. The analysis does not indicate the existence of traveling hyperbolic-secant solitons of the Schrödinger type.

DOI: 10.1103/PhysRevE.66.046610 PACS number(s): 42.79.Gn, 42.70.Qs, 42.25.Bs, 42.65.Wi

I. INTRODUCTION

A coupled-resonator optical waveguide (CROW) [1–3] is composed of a periodic array of isolated structural elements (e.g., high-Q resonators such as defects in photonic crystals—see Fig. 1) weakly coupled to one another. Such waveguides are naturally described by the tight binding approximation [4], in direct correspondence with the description of electrons in a strong periodic potential in solid state physics [5]. Experimental demonstrations of the CROW concept and corroboration of the analytical model were recently presented [6,7]. Prior to the introduction of the generic CROW family of waveguides, the tight binding formalism was applied to the description of deep superstructure gratings [8].

The dispersion relationship in CROWs is intrinsically nonlinear, but the propagation of localized excitations, i.e., optical pulses can be characterized nonperturbatively to all orders of dispersion [9]. This is a somewhat surprising result, and it leads to a description of the distortion that results from the nonlinear dispersion relationship. Weighted sums of Bessel functions take the role of cosines in the Fourier-series decomposition of the propagating field [10].

In optical fibers and similar waveguides, the effects of (anomalous) group-velocity dispersion can be exactly balanced by the self-phase modulation induced by the Kerr effect, an intensity-dependent change in the refractive index of the material. This is the basis for the formation of the (fundamental) Schrödinger soliton in optical fibers, for instance. Here, we investigate the Kerr effect in coupled-resonator waveguides, with particular emphasis on determining whether self-phase modulation can compensate for the distortion consequent of the nonlinear dispersion relationship.

Such solutions would lead to the existence of envelopes that can exist or propagate without distortion in CROWs as eigensolutions of a nonlinear propagation equation (solitary waves and solitons). Whether such soliton pulse shapes exist or not, from a practical viewpoint (since the material dispersion also plays a role), nonlinear propagation in such waveguides can be controlled, as in the linear cases analyzed thus far, by choosing (or changing) the structural properties of the waveguide, e.g., inter-resonator spacing, overlap integrals between adjacent resonator eigenmodes, and the Fourier spectrum of the initial excitation. Coupled-resonator waveguides therefore offer a wider range of design possibilities for the realization of all-optical information processing devices than available thus far.

In considering the various choices in which to expand the field, we choose the propagating Bloch wave solutions of the CROWs without optical nonlinearities, which are derived from the tight-binding approximation [1–3]. We take the field to be a superposition of such waves with slowly (time-) varying coefficients. This expansion has the merit that in the absence of nonlinearities, each field in this expansion is an eigenmode.

II. WAVEGUIDE MODES AND LINEAR PROPAGATION

We assume that the structural elements comprising the periodic waveguide of length L, e.g., defects in a 2D photonic crystal slab with index confinement in the out-of-plane direction, are identical and lie along the z axis (unit vector $\mathbf{e}_z$) separated by a distance R. Together with its time-evolution factor, the waveguide mode of the linear waveguide (an eigenmode of a time-independent Hamiltonian) $\boldsymbol{\phi}_k(\mathbf{r})$ at a particular propagation constant k is written as a linear combination of the individual eigenmodes $\mathbf{E}_{\text{res}}(\mathbf{r})$ of

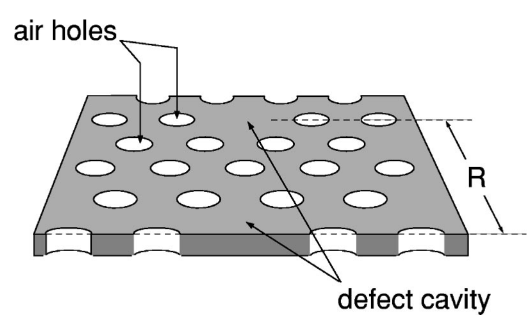

FIG. 1. Schematic of an infinitely-long 1D CROW with periodicity R consisting of defect cavities embedded in a 2D photonic crystal. The dielectric material in the defect cavities exhibits the nonlinear Kerr effect, i.e., its refractive index is modified by the optical intensity.

*Electronic address: shayan@caltech.edu

the elements that comprise the structure [3,5],

$$e^{-i\omega_k t}\boldsymbol{\phi}_k(\mathbf{r}) = e^{-i\omega_k t}\sum_n \exp(inR\,\mathbf{k}\cdot\mathbf{e}_z) \times \mathbf{E}_{\text{res}}(\mathbf{r}-nR\mathbf{e}_z), \tag{1}$$

where the summation over n runs over the $N=L/R$ structural elements and we consider only a single bound state in each individual element. As expected, Eq. (1) has the Bloch form [5].

The dispersion relationship for a CROW mode around a central wave number k_0 is [1]

$$\omega_{k_0+K} = \Omega(1-\Delta\alpha/2) + \Omega\kappa\cos[(k_0+K)R] \equiv \omega_0 + \Delta\omega\cos[(k_0+K)R], \tag{2}$$

where Ω is the eigenfrequency of the individual resonators. In Eq. (2), $\Delta\alpha$ and κ are overlap integrals involving the individual resonator modes and the spatial variation of the dielectric constant,

$$\Delta\alpha = \int d^3\mathbf{r}[\epsilon_{\text{wg}}(\mathbf{r}-R\mathbf{e}_z) - \epsilon_{\text{res}}(\mathbf{r}-R\mathbf{e}_z)]|\mathbf{E}_{\text{res}}(\mathbf{r})|^2,$$

$$\kappa = \int d^3\mathbf{r}[\epsilon_{\text{res}}(\mathbf{r}-R\mathbf{e}_z) - \epsilon_{\text{wg}}(\mathbf{r}-R\mathbf{e}_z)] \times \mathbf{E}_{\text{res}}(\mathbf{r})\cdot\mathbf{E}_{\text{res}}(\mathbf{r}-R\mathbf{e}_z), \tag{3}$$

where ϵ_{res} is the dielectric constant of the individual resonators, and ϵ_{wg} is the dielectric constant of the waveguide. We restrict the range of K to the first Brillouin zone, $|K|R<\pi$. One may usually assume for convenience (as in [3]) that $k_0R=2m\pi$, for some integer m, but in this paper we will work in the general case, unless stated otherwise.

The field describing a pulse $\boldsymbol{\mathcal{E}}(\mathbf{r},t)$ is written as a superposition of waveguide modes $\boldsymbol{\phi}_k(\mathbf{r})$ within the Brillouin zone, and using Eq. (2),

$$\boldsymbol{\mathcal{E}}(\mathbf{r},t) \approx \int \frac{dk}{2\pi} e^{-i\omega_k t} c_k \boldsymbol{\phi}_k(\mathbf{r}) = e^{-i\omega_0 t}\int_{-\pi/R}^{\pi/R}\frac{dK}{2\pi}[e^{-i\Delta\omega t\cos[(k_0+K)R]} c_{k_0+K}\boldsymbol{\phi}_{k_0+K}(\mathbf{r})]. \tag{4}$$

The boundary conditions that arise in pulse propagation problems typically specify a pulse shape at the $\mathbf{r}=0$ cross section of the waveguide and centered at the optical frequency ω_0,

$$\boldsymbol{\mathcal{E}}(\mathbf{r}=0,t) = e^{-i\omega_0 t}E(z=0,t)\hat{\mathbf{u}}, \tag{5}$$

where $\hat{\mathbf{u}}$ is a unit-magnitude vector that describes the vectorial nature of the field at $\mathbf{r}=0$.

Finding the coefficients c_{k_0+K} in Eq. (4) based on the boundary condition, Eq. (5), is relatively simple when we approximate the dispersion relationship, Eq. (2), as linear [11]. It is also possible to solve this problem without making this approximation [9], and this is the framework which we will use to analyze *nonlinear* propagation in this paper. The form of the dispersion relationship, Eq. (2), is the signature of all tight-binding models based on nearest neighbor interactions [5], and it is important to keep the full form in describing those nonlinear phenomena which depend on a balance between nonlinear and dispersive phase-modulation effects in a single wave form, e.g., solitons.

III. FORMULATION OF THE NONLINEAR PROPAGATION PROBLEM

Since nonlinear phenomena such as the Kerr effect change the relative weights of the eigenmodes Eq. (1) as the wave form evolves with time, we introduce a time dependency in the superposition coefficients c_k appearing in Eq. (4). [We assume that the Hilbert space of solutions is spanned by the eigenmode set Eq. (1).] We write

$$\boldsymbol{\mathcal{E}}(\mathbf{r},t) = e^{-i\omega_0 t}\int_{-\pi/R}^{\pi/R}\frac{dK}{2\pi}[e^{-i\Delta\omega t\cos[(k_0+K)R]} \times c_{k_0+K}(t)\,\boldsymbol{\phi}_{k_0+K}(\mathbf{r})]. \tag{6}$$

Equation (6) leads to a differential equation describing the evolution of the time-varying coefficients, driven by the nonlinear (Kerr) polarization,

$$\frac{dc_{k_0+K}(t)}{dt} = i\gamma\int\int_{-\pi/R}^{\pi/R}\frac{dK_1}{2\pi}\frac{dK_2}{2\pi}\exp[-i\kappa\Omega t \times\{-\cos[(k_0+K_1)R]+\cos[(k_0+K_2)R] + \cos[(k_0+K_3)R]-\cos[(k_0+K)R]\}] \times c_{k_0+K_1}(t)^* c_{k_0+K_2}(t) c_{k_0+K_3}(t), \tag{7}$$

where $K_1+K=K_2+K_3$ and γ is the nonlinearity coefficient in the CROW geometry, described in the Appendix.

IV. LINEAR AND NONLINEAR EVOLUTION

A. Linear case

We first note that in the linear case, when $\gamma=0$, the solution of Eq. (7) is trivial: $c_{k_0+K}(t)=c_{k_0+K}(t=0)$ as would be expected on physical grounds: The waveguide modes, being orthogonal, are uncoupled. The value of $c_{k_0+K}(t=0)$ may be evaluated for an arbitrary input pulse shape as described in [9]:

$$c_{k_0+K}(0) = \frac{1}{\phi_{k_0+K}(0)}\left\{\sum_{n=1}^{\infty}\frac{2nR}{b_n}\left(\int_0^{\infty}\frac{dt'}{t'} \times[E(z=0,t'/\Delta\omega)-E(0,0)]J_n(t')\right) \times\cos(nKR) + c_{k_0}\phi_{k_0}(0)\right\}. \tag{8}$$

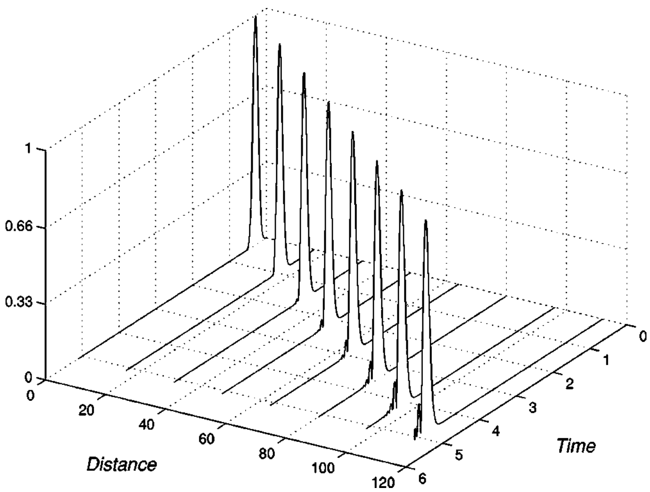

FIG. 2. Temporal evolution of a Gaussian envelope at specific distances inside a CROW, showing the effects of dispersive propagation. "Distance" is normalized to R, the inter-resonator spacing. "Time" is normalized to $1/\Delta\omega$. The vertical axis represents $|\mathcal{E}(z,t=0)|$ normalized to its maximum value. At greater depths, the peak of the envelope arrives at a later time, and ripples in the trailing edge indicate higher-order distortion.

For waveguides composed of a large number of resonators, we have derived a simpler expression [10],

$$c_{k_0+K}(0)=\frac{2R}{\phi_{k_0+K}(0)}\sum_{n=0}^{\infty}\frac{\beta_n}{2}\cos[n(k_0+K)R], \qquad (9)$$

where

$$\beta_n=\begin{cases} R^{-1}c_{k_0}|\phi_{k_0}(0)|, & n=0,\\ i^{-n}[E(z=0,n/\Delta\omega)-E(0,0)], & n\geqslant 1,\end{cases} \qquad (10)$$

and c_{k_0} is given by Parseval's relationship.

An example of the propagation of a (temporal) Gaussian pulse through a coupled resonator waveguide with Gaussian functions describing the spatial distribution of the eigenmodes is shown in Fig. 2. As may be expected from the dispersion relationship, distortion accumulates with distance, and is manifest in the oscillatory structure on the trailing edge of the pulse. The crest of the envelope travels with an approximate group velocity $\Delta z/\Delta t \lesssim \Delta\omega R$; an exact group velocity is not defined since the dispersion relationship is nonlinear.

B. Nonlinear case

We separate the amplitude and phase of $c_{k_0+K}(t)$ as

$$c_{k_0+K}(t)=A_{k_0+K}(t)\exp[i\phi_{k_0+K}(t)]. \qquad (11)$$

We will look for solutions that retain their shape, i.e., $dA/dt=0$. Substituting Eq. (11) into Eq. (7) and separating the real and imaginary parts, we obtain a pair of equations,

$$\frac{dA_{k_0+K}}{dt}=-\gamma\int\int\frac{dK_1}{2\pi}\frac{dK_2}{2\pi}A_{k_0+K_1}A_{k_0+K_2}A_{k_0+K_3}\sin\Phi, \qquad (12)$$

$$\frac{d\phi_{k_0+K}}{dt}=\frac{\gamma}{A_{k_0+K}}\int\int\frac{dK_1}{2\pi}\frac{dK_2}{2\pi}A_{k_0+K_1}A_{k_0+K_2}\times A_{k_0+K_3}\cos\Phi, \qquad (13)$$

where Φ is defined as

$$\Phi\equiv-\{\phi_{k_0+K_1}-\kappa\Omega t\cos[(k_0+K_1)R]\}+\{\phi_{k_0+K_2}-\kappa\Omega t\times\cos[(k_0+K_2)R]\}+\{\phi_{k_0+K_3}-\kappa\Omega t\cos[(k_0+K_3)R]\}-\{\phi_{k_0+K}-\kappa\Omega t\cos[(k_0+K)R]\}. \qquad (14)$$

Based on Eq. (12), the A's will be independent of t if $\sin\Phi\equiv 0$ for all t. This implies that $\cos\Phi=1$, and based on Eq. (13), we take ϕ_{k_0+K} to be a linear function of t,

$$\phi_{k_0+K}(t)=a+bt+\kappa\Omega t\cos[(k_0+K)R], \qquad (15)$$

where a and b are constants independent of t and K. We drop the constant a which represents a fixed phase that can be absorbed into the initial conditions. Substituting this form for $\phi_{k_0+K}(t)$ into Eq. (13), we get

$$b+\kappa\Omega\cos[(k_0+K)R]=\frac{\gamma}{A_{k_0+K}}\int\int_{-\pi/R}^{\pi/R}\frac{dK_1}{2\pi}\frac{dK_2}{2\pi}A_{k_0+K_1}A_{k_0+K_2}A_{k_0+K_3}. \qquad (16)$$

We will discuss numerical techniques to the solution of Eq. (16) in a separate paper; here, we discuss a particular regime in which there exist stationary solutions.

V. DISCUSSION: TIME-INVARIANT EVOLUTION AND THE NONLINEAR SCHRÖDINGER EQUATION

In this section, we will use the results from Sec. IV B to discuss in what regime the CROW admits solutions of the Schrödinger soliton form, i.e., the hyperbolic secant. The basic physics lie in a balance between the phase modulation effects of the Kerr effect and (anomalous) group-velocity dispersion (GVD). The GVD term in the nonlinear Schrödinger equation appears as the coefficient of a second derivative term, which in the Fourier domain with the Fourier (frequency) variable K, translates to multiplication by $(iK)^2$.

In Eq. (16), if we assume that k_0R is a multiple of 2π and $|KR|\ll 1$, then we may write $\cos[(k_0+K)R]\approx 1-(KR)^2/2$, which is the desired effective GVD term. Observe from the dispersion relationship, Eq. (2), that ω_{k_0+K} is a quadratic function of K only at the edges of the Brillouin zone—where $d\omega_{k_0+K}/dK$ vanishes, i.e., the group velocity is zero. We expect, therefore, that the solutions of Eq. (16) in this regime will be *stationary*, describing a localized state that is frozen in its initial ($t=0$) spatial distribution and does not propagate along the waveguide.

Using this approxmation, Eq. (16) becomes

$$b+\kappa\Omega=\kappa\Omega\frac{(KR)^2}{2}+\frac{\gamma}{A_{k_0+K}}\int\int_{-\pi/R}^{\pi/R}\frac{dK_1}{2\pi}\frac{dK_2}{2\pi}\times A_{k_0+K_1}A_{k_0+K_2}A_{k_0+K_3}. \tag{17}$$

We assume that the A's are defined to be zero outside the regions of integration $-\pi/R$ and π/R so that the limits of integration can be taken as $-\infty$ to ∞. Equation (17) may then be solved [12],

$$A_{k_0+K}=A^{(0)}_{k_0+K}\mathrm{sech}(K/\bar{K}), \tag{18}$$

where $\bar{K}$ is a spectral width parameter whose relevance will become clear in the following discussion. Substituting Eq. (18) into Eq. (17), we get

$$b+\kappa\Omega=\kappa\Omega\frac{(KR)^2}{2}+2[A^{(0)}_{k_0+K}]^2\frac{\gamma}{(2\pi R)^2}\times\left[(KR)^2+\left(\frac{\pi\bar{K}R}{2}\right)^2\right]. \tag{19}$$

If b is to be independent of K, then we need

$$A^{(0)}_{k_0+K}=\sqrt{-(2\pi R)^2\frac{\kappa\Omega}{4\gamma}}. \tag{20}$$

Since the left-hand side represents a real and positive number, we require that κ as defined in Eq. (3) be a negative number (as is physically expected from the meaning of ϵ_{wg} and ϵ_{res}). This is equivalent to anomalous dispersion in optical fibers and similar waveguides.

Using Eq. (15) and Eq. (20) in Eq. (11), we write the final expression for $c_{k_0+K}(t)$,

$$c_{k_0+K}(t)=c_{k_0+K}(0)\exp\{-i\kappa\Omega t[1+\pi^2(\bar{K}R)^2/8-\cos(KR)]\}, \tag{21}$$

where

$$c_{k_0+K}(0)\equiv 2\pi R\sqrt{-\frac{\kappa\Omega}{4\gamma}}\mathrm{sech}(K/\bar{K}),\quad |KR|\leqslant\pi. \tag{22}$$

The field described by these coefficients is

$$\boldsymbol{\mathcal{E}}(\mathbf{r},t)=e^{-i\omega_0 t}e^{-i\kappa\Omega t[1+\pi^2(\bar{K}R)^2/8]}\times\int_{-\pi/R}^{\pi/R}\frac{dK}{2\pi}c_{k_0+K}(0)\boldsymbol{\phi}_{k_0+K}(\mathbf{r}). \tag{23}$$

In light of Eq. (22), the integral on the second line of Eq. (23) is not expressible in a simpler form. However, if $\bar{K}R\lesssim 1$, the hyperbolic secant function decays rapidly, and the limits of integration may be changed to $(-\infty,\infty)$. The integral then can be evaluated easily—the Fourier transform of a hyperbolic secant is itself a hyperbolic secant function. We derive the approximation

$$\boldsymbol{\mathcal{E}}(\mathbf{r},t)\approx e^{-i\omega_0 t}e^{-i\kappa\Omega t[1+\pi^2(\bar{K}R)^2/8]}\sqrt{-\frac{\kappa\Omega}{4\gamma}}\times\pi\bar{K}R\sum_n\mathrm{sech}\left(\frac{\pi\bar{K}}{2}nR\right)\mathbf{E}_{\mathrm{res}}(\mathbf{r}-nR\hat{z}). \tag{24}$$

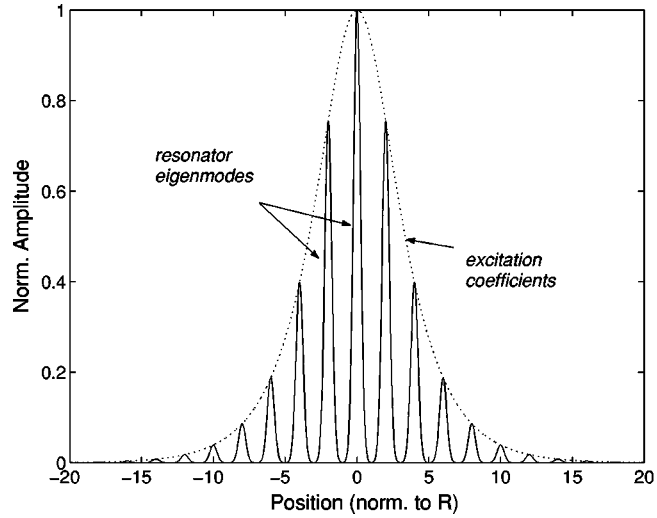

FIG. 3. An approximate super-resonant field distribution, assuming that the individual resonator eigenmodes are Gaussians. "Position" is normalized to R, the inter-resonator spacing and the ordinate represents $|\mathcal{E}(z,t=0)|$ normalized to its maximum value. The dotted line is an envelope—a hyperbolic secant—connecting the excitation coefficients multiplying the individual resonator eigenmodes. We have used $\pi\bar{K}/2=\pi/(4R)$.

The modulus of the amplitude $|\mathcal{E}(z,t=0)|$ normalized to its maximum value (in this approximation) is plotted in Fig. 3. Values of the hyperbolic secant function in Eq. (24) at nR (which has the dimensions of length) are the weights of the individual resonator eigenmodes. In this approximation, the envelope of these weights is a hyperbolic secant function whose width is inversely proportional to $\bar{K}$.

As we had expected from physical arguments, the envelope of $\boldsymbol{\mathcal{E}}(\mathbf{r},t)$ is a stationary state that is independent of time: its spatial distribution at $t=0$ is maintained for all subsequent t. This is consistent with the observation that although the group velocity dispersion coefficient is nonzero, the group velocity itself is zero. We call the stationary state a super-resonant field since it is formed in a waveguide that itself comprises the coupling of individual (stationary) resonator modes. There are two requirements for such a solution: (1) the slowly varying assumption, which simplifies Eq. (23) to Eq. (24) and (2) the *necessary* condition that k_0R is an integer multiple of 2π.

Using Eq. (A6), Eq. (24) may be rephrased as an expression for the individual-resonator coefficients $a_n(t)$,

$$a_n(z_n)=\left[\sqrt{-\frac{\kappa\Omega}{4\gamma}}\pi\bar{K}R\right]e^{-i\mu t}\mathrm{sech}\left(\frac{\pi\bar{K}}{2}z_n\right), \quad (25)$$

where $\mu\equiv\kappa\Omega[1+\pi^2(\bar{K}R)^2/8]$ is a constant frequency detuning and $z_n\equiv nR$ is a discretization of the spatial axis.

Christodoulides and Efremedis [13] have analyzed this problem using the a_n coefficients [see Eq. (A6)]. In contrast with our analysis, they predict the existence of moving hyperbolic-secant solitons as well as stationary solitons. Their envelopes, similar to Eq. (25), propagate with a group velocity $v\equiv-\kappa\Omega R\sin q$, where q is a parameter that appears in an assumed ansatz. The two solutions agree only when $v=0$ (and therefore q is not an undetermined parameter), and this solution does not propagate along the waveguide. Using finite-difference time-domain calculations with a nonlinear polarization term accounting for the Kerr effect in coupled-defect waveguides, Iliew *et al.* [14] have shown that there exist stable and localized (nonpropagating) envelopes, similar to nonlinearly localized modes or discrete solitons as found in discrete systems [15]. Propagating soliton-type envelopes have not yet been found in numerical simulations.

The super-resonant mode in a CROW composed of high-Q resonators can have a long lifetime, since it decays with the time constant associated with the quality factor of the isolated individual resonators [2] rather than the time constant associated with the coupling between high-Q resonators and external waveguides. In addition, an optical pulse (with nonzero group velocity) traveling down the waveguide can be made to interact with such a static distribution; their interaction can be enhanced using quasi-phase-matching (grating) techniques [16]. This leads to the possibility of the application of these localized states for optical switching and routing.

VI. CONCLUSION

We have investigated the effects of the optical Kerr nonlinearity in coupled-resonator optical waveguides (CROWs) with regard to the propagation of optical pulses. In particular, we have shown that there exists a stationary field distribution of the hyperbolic secant form which balances the effects of group velocity dispersion and the Kerr self-phase modulation. This field distribution is closely related to the family of gap solitons in periodic structures, but remains frozen in space with zero group velocity.

On the fundamental waveguiding aspects of the Kerr effect, we have derived an equation closely related to the frequency representation of the nonlinear Schrödinger equation. Our approach may also be used to describe any nonlinear wave-mixing phenomena in all electromagnetic tight-binding models.

ACKNOWLEDGMENTS

The authors thank D.S. Cohen of the California Institute of Technology for useful discussions. This work was funded by the US Office of Naval Research and the US Air Force Office of Scientific Research.

APPENDIX: EVOLUTION OF THE TIME-VARYING COEFFICIENTS

We substitute Eq. (6) which describes the field in the waveguide in terms of the time-varying coefficients $c_{k_0+K}(t)$ into Maxwell's equations written with an explicit nonlinear polarization term describing the Kerr effect,

$$\nabla\times\nabla\times\boldsymbol{\mathcal{E}}(\mathbf{r},t)-\mu\epsilon_{\mathrm{wg}}(\mathbf{r})\frac{\partial^2}{\partial t^2}\mathcal{E}(\mathbf{r},t)=\mu\frac{\partial^2}{\partial t^2}\mathbf{P}_{\mathrm{NL}}(\mathbf{r},t), \quad (A1)$$

where

$$\mathbf{P}_{\mathrm{NL}}(\mathbf{r},t)=\frac{3}{4}\epsilon_0\chi^{(3)}|\boldsymbol{\mathcal{E}}(\mathbf{r},t)|^2\boldsymbol{\mathcal{E}}(\mathbf{r},t) \quad (A2)$$

in the instantaneous response approximation.

In simplifying the terms, we use the normalization [16]

$$M\sum_m\int d\mathbf{r}\epsilon_{\mathrm{wg}}(\mathbf{r})|\mathbf{E}_{\mathrm{res}}(\mathbf{r}-mR\mathbf{e}_z)|^2=1, \quad (A3)$$

where the CROW waveguide comprises M resonators.

If we assume that $c_{k_0+K}(t)$ varies slowly over time intervals $\sim O(2\pi/\omega_0)$, as is usually the case, then we obtain Eq. (7). The nonlinearity coefficient is defined as

$$\gamma=2n_0n_2\epsilon_0\omega_0\int d\mathbf{r}\sum_m|\mathbf{E}_{\mathrm{res}}(\mathbf{r}-mR\mathbf{e}_z)|^4, \quad (A4)$$

using the relationship $3\chi^{(3)}/8=n_0n_2$ [17], and we have ignored the dispersion (variation in ω) of γ.

Equation (7) is equivalent to the differential equation

$$i\frac{da_n}{dt}+\frac{\Delta\alpha}{2}\Omega a_n-\frac{\kappa}{2}\Omega(a_{n+1}+a_{n-1})+\gamma|a_n|^2a_n=0, \quad (A5)$$

obtained by Christodoulides and Efremidis [13] for a related set of coefficients, $a_n(t)$, where

$$a_n(t)=\int_{-\pi/R}^{\pi/R}\frac{dK}{2\pi}c_{k_0+K}(t)\exp[in(k_0+K)R] \times\exp\left[i\left\{\frac{\Delta\alpha}{2}\Omega-\kappa\Omega\cos[(k_0+K)R]\right\}t\right]. \quad (A6)$$

The notational correspondence from our paper to theirs is $(\Delta\alpha/2)\Omega\mapsto\Delta\omega$ and $-(\kappa/2)\Omega\mapsto c$. The a_n's are the coefficients that appear in the expansion of the field in terms of individual resonator modes, rather than the waveguide modes. It is easily verified that substituting the plane-wave ansatz $a_n=\exp\{i[(\Omega-\omega_{k_0+K})t-(k_0+K)nR]\}$ in Eq. (A5) with $n_2=0$ leads to the dispersion relationship, Eq. (2).

This basis set consisting of individual resonator modes is not an orthonormal set, unlike our waveguide modes $\{\boldsymbol{\phi}_k(\mathbf{r})\exp(i\omega_k t)\}$.

[1] A. Yariv, Y. Xu, R. Lee, and A. Scherer, Opt. Lett. **24**, 711 (1999).
[2] Y. Xu, R. Lee, and A. Yariv, J. Opt. Soc. Am. B **17**, 387 (2000).
[3] S. Mookherjea and A. Yariv, IEEE J. Sel. Top. Quantum Electron **8**, 448 (2002).
[4] N. Stefanou and A. Modinos, Phys. Rev. B **57**, 12127 (1998).
[5] N. Ashcroft and N. Mermin, *Solid State Physics* (Harcourt, Fort Worth, 1976).
[6] M. Bayer, I. Gutbrod, A. Forchel, T. Reinecke, P. Knipp, R. Werner, and J. Reithmaier, Phys. Rev. Lett. **83**, 5374 (1999).
[7] S. Olivier, C. Smith, M. Rattier, H. Benisty, C. Weisbuch, T. Krauss, R. Houdré, and U. Oesterlé, Opt. Lett. **26**, 1019 (2001).
[8] C. de Sterke, Phys. Rev. E **57**, 3502 (1998).
[9] S. Mookherjea and A. Yariv, Phys. Rev. E **65**, 056601 (2002).
[10] S. Mookherjea, D. Cohen, and A. Yariv, Opt. Lett. **27**, 933 (2002).
[11] S. Mookherjea and A. Yariv, Opt. Express **9**, 91 (2001).
[12] C. Pask and A. Vatarescu, J. Opt. Soc. Am. B **3**, 1018 (1986).
[13] D. Christodoulides and N. Efremidis, Opt. Lett. **27**, 568 (2002).
[14] R. Iliew, U. Peschel, C. Etrich, and F. Lederer, in *Conference on Lasers and Electro-optics, OSA Technical Digest, Postconference Edition*, OSA Trends in Optics and Photonics (TOPS), (Optical Society of America, Washington, DC, 2002), Vol. 73 pp. 191–192.
[15] F. Lederer, S. Darmanyan,and A. Kobyakov, *Spatial Solitons* (Springer, New York, 2001), pp. 269–292.
[16] S. Mookherjea and A. Yariv, Phys. Rev. E **64,**, 066602 (2001).
[17] G. Agrawal, *Nonlinear Fiber Optics* (Academic, New York, 1989).